The Fundamental Group of a Surface

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https://sites.google.com/a/dons.usfca.edu/surfacegroupsbasics/

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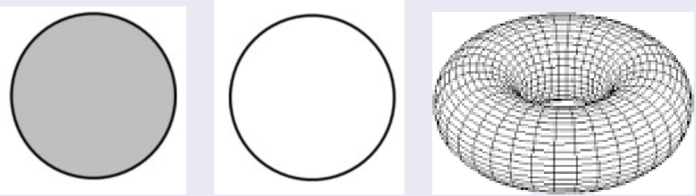
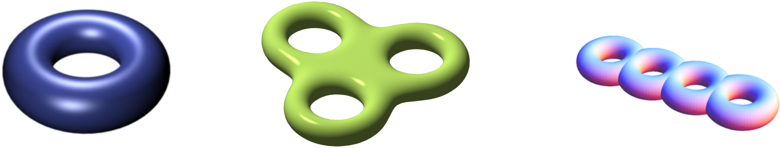
Introduction

Surface Groups provide a cornerstone for the study of Algebraic Topology, which develops methods for understanding and analyzing topological spaces through algebraic means.

The Fundamental Group, π_1

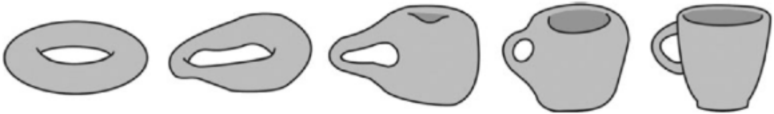


"Simple Descriptions": genus (holes) $\longleftrightarrow$ fundamental group



Motivation

Problem: Just like a group, a single topological space can wear an infinite number of disguises!



In order to study surfaces algebraically, we need to be able to tell what kind of space we're looking at.

Solution: The Fundamental Group is a topological invariant! That is, two homeomorphic topological spaces have isomorphic fundamental groups.

History

- (1752) Euler Characteristic
- (1841) Abel defines *genus*
- (1851) Riemann relates the Euler Characteristic to genus in his development of "Riemann Surfaces"
- (1863) Mobius classifies all (orientable, funny enough) closed surfaces in $\mathbb{R}^3$ by genus / Euler Characteristic

Breakthrough: (1871) Betti numbers, which equate to the genus / EC of a surface, could be used on surfaces in higher dimensions that were previously unclassified!

Are Betti numbers enough to classify *all* surfaces?
Cue Poincaré...

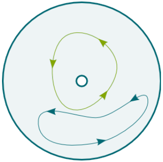
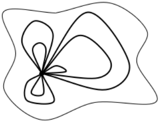
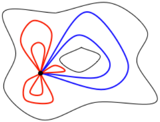
Analysis Situs

- (1895) *Analysis situs* $\leftrightarrow$ geometry of position $\leftrightarrow$ topology
- ". . . geometry is the art of reasoning well from badly drawn figures; however, these figures, if they are not to deceive us, must satisfy certain conditions; the proportions may be grossly altered, but the relative positions of the different parts must not be upset." (Stillwell's Translation)
- In his investigation, Poincaré developed the Fundamental Group in order to test the rigor of Betti numbers
- Punchline: Poincaré found several surfaces whose Betti numbers were the same but their fundamental groups were different!

What is a Fundamental Group?

Some quick and dirty definitions...

- **Loop** (X Path-Connected): $\alpha(s) : [0,1] \rightarrow X$
  - $\alpha(0) = \alpha(1)$
- **Homotopy**: $H(s,t) : [0,1]x[0,1] \rightarrow X; \quad H(s,t) = H_t(s)$
  - H takes one loop α, and morphs it into β
  - s is the same as before, t indicates the interim loops $H_0 = \alpha$, $H_1 = \beta$
  - illustrate with string analogy on board
  - If such a function exists, α is **homotopic** to β
- When are two loops in X not homotopic?

- **Homotopy Classes, $<\alpha>$**

The Trivial Fundamental Group

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- **Homotopy Classes of a space, X, form the Fundamental Group, $\pi(X)$**
  - Operation: concatenation $\longleftrightarrow$ composition
  - Assosciativity? $<\alpha> \bullet <\beta> = <\alpha \bullet \beta> = <\alpha \circ \beta>$
  - Identity $<\epsilon>$? basepoint
  - Inverses $(<\alpha>)^{-1}$? $<\alpha^{-1}>$
- **Loops in the disk** illustrate on board
  - $H(s,t) = H_t(s) = \alpha(s) + t(x_0 - \alpha(s))$
  - How many classes do we have? Just one!
  - $\pi_1(D^2) \cong \{\epsilon\}$
  - applies to all convex subsets of Euclidean space
- **Loops in the sphere** illustrate on board
  - $\pi_1(S^2) \cong \{\epsilon\}$
- **Simply Connected**

S1 : The Circle

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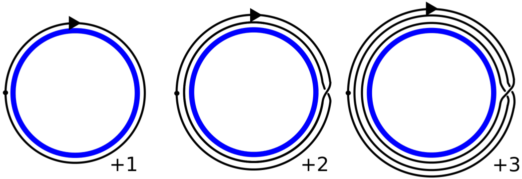
- $\alpha, \beta \in S^1$, when do we have $< \alpha > \neq < \beta >$?

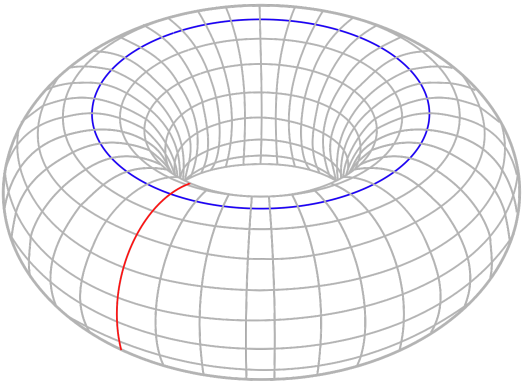


- **Define** $\bar{\alpha} : [0,1] \to \mathbb{R}$ to measure the net angle
  - $\alpha(t) = (cos(\bar{\alpha}(t)), sin(\bar{\alpha}(t)))$, $\bar{\alpha}(0) = 0$
  - What is $\bar{\alpha}(1)$? $(2\pi)k$ for some $k \in \mathbb{N}$
- **Define** $deg\alpha = \frac{1}{2\pi}\bar{\alpha}(1)$
  - **Theorem** : $\alpha, \beta \in S^1$ homotopic $\iff$ $deg\alpha = deg\beta$
  - **Define** $deg < \alpha >= deg\alpha$
  - **lemma** : $(< \alpha > \bullet < beta >) = deg < \alpha > + deg < \beta >$

What does the group look like? $\pi_1(S^1) \cong \mathbb{Z}$

T2 : The Torus

$T^2 = S^1 x S^1 \rightarrow$



$\pi_1(T^2) = \pi_1(S^1 x S^1) = \pi_1(S^1) x \pi_1(S^1) = \mathbb{Z} x \mathbb{Z}$

Bibliographication

Final Project

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References

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2. *Papers on Topology:* Analysis Situs *and Its Five Supplements*, Henri Poincaré, Translated by John Stillwell, 2009, PDF: http://www.maths.ed.ac.uk/ aar/papers/poincare2009.pdf
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